TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 7, July 1995

Π_1^1 FUNCTIONS ARE ALMOST INTERNAL

BOŠKO ŽIVALJEVIĆ

ABSTRACT. In *Analytic mappings on hyperfinite sets* [Proc. Amer. Math. Soc. 2 (1993), 587–596] Henson and Ross asked for what hyperfinite sets S and T does there exists a bijection f from S onto T whose graph is a projective subset of $S \times T$? In particular, when is there a Π_1^1 bijection from S onto T? In this paper we prove that given an internal, bounded measure μ, any Π_1^1 function is $L(\mu)$ a.e. equal to an internal function, where $L(\mu)$ is the Loeb measure associated with μ. It follows that if two Π_1^1 subsets S and T of a hyperfinite set X are Π_1^1 bijective, then S and T have the same measure for every uniformly distributed counting measure μ. When S and T are internal it turns out that any Π_1^1 bijection between them must already be Borel. We also prove that if a Π_1^1 graph in the product of two hyperfinite sets X and Y is universal for all internal subsets of Y, then $|X| \geq 2^{|Y|}$, which is a partial answer to Henson and Ross's Problem 1.5. At the end we prove some standard results about the projections and a structure of co-proper K-analytic subsets of the product of two completely regular Hausdorff topological spaces with open vertical sections. We were able to prove the above results by revealing the structure of Π_1^1 subsets of the products $X \times Y$ of two internal sets X and Y, all of whose Y-sections are $\Sigma_1^0(\kappa)$ sets.

1. INTRODUCTION

In a recent paper [3] C.W. Henson and D. Ross proved, using Choquet's capacitability theorem, that, in the setting of Descriptive Set Theory of Hyperfinite Sets, any function f whose graph is a Σ_1^1 subset of the product $X \times Y$ of two internal sets X and Y can be well approximated by internal functions. Given any internal, $*$-finitely additive, bounded measure μ there exists an internal function φ such that $f = \varphi$, $L(\mu)$-a.e., where $L(\mu)$ is the Loeb measure associated with μ. This in turn shows that, for hyperfinite X and Y and μ an internal counting measure, injective Σ_1^1 functions preserve $L(\mu)$ measure (the image of a $L(\mu)$ measurable set is a $L(\mu)$ measurable set of the same measure). In particular, if two hyperfinite sets S and T are Borel bijective, then $|S|/|T| \approx 1$ and, conversely, any two hyperfinite sets S and T satisfying $|S|/|T| \approx 1$ are Borel bijective.

It is natural to ask if the approximation of f is still possible if the function f is a member of some level of the finite projective hierarchy (Problem 1.4. in

Received by the editors August 6, 1993 and, in revised form, May 20, 1994; originally communicated to the *Proceedings of the AMS* by Andreas R. Blass.

1991 *Mathematics Subject Classification*. Primary 03H04, 03E15; Secondary 04A15, 54H05, 54J05.

[3]) other than Σ_1^1. In particular, is it true that if f is Π_1^1, then f is almost equal to an internal function? In this paper we consider only the latter problem and, in fact, ask and answer a slightly more general question than the one stated above. We prove that any Π_1^1 graph with internal vertical sections can be well approximated by internal graphs and any Π_1^1 graph with $\Sigma_1^0(\kappa)$ vertical sections can be well approximated by Σ_1^0 graphs (for definitions see below). In fact, as we prove in Theorem 1, any such graph can be represented as a union of countably many restrictions of internal graphs to (not necessarily disjoint) Π_1^1 sets, possesses a Π_1^1 uniformization, and has a Π_1^1 domain. In particular, any Π_1^1 functions is a union of restrictions of internal functions to disjoint Π_1^1 sets. It follows that if a Π_1^1 function has internal (or more generally Σ_1^1) domain, then it must be Borel (the fact already proved in [5], Corollary 4.5 b)).

We also give a partial answer to Problem 1.5 in [3] which asked for the connection between the internal cardinalities of two hyperfinite sets X and Y for which there exists a Σ_1^1 graph $\Gamma \subseteq X \times Y$ universal for all Σ_1^1 subsets of Y. We show (Corollary 6) that if Γ is universal for all internal subsets of Y, then the quotient $|X|/2^{|Y|}$ is greater than or infinitely close to 1.

At the end of the paper we apply the above results in the standard descriptive set theory of general topological spaces. We prove that a co-K-analytic graph G in the product of a proper K-Lusin space X and a proper K-Lusin and locally compact space Y, all of whose Y-sections are open sets in Y, has a co-K-analytic domain and is almost equal (with respect to a given Radon bounded measure μ in X) to a σ-compact graph in $X \times Y$ (for definitions see below). Also, the set of all points in X where the vertical section of G is compact and nonempty is always μ measurable.

Acknowledgments. I would like to thank Professor Ward Henson and Professor David Ross who made the early preprint of their paper [3] available to me long before the article was published. Also, I would like to thank Hermann Render for sharing his recent results with me and to the referee for a thorough reading of the paper.

2. Definitions and terminology

By a *Borel* set in an internal set X we mean a member of the least σ-algebra generated by internal subsets of X. The *Borel hierarchy* of Σ_α^0 and Π_α^0 $(\alpha < \omega_1)$ subsets of X is introduced as usual; internal subsets of X are called Σ_0^0 or Π_0^0 sets, and a set is Π_α^0 (Σ_α^0) if it is a countable intersection (union) of sets A_n with each A_n being a Σ_β^0 (Π_β^0) set for some $\beta < \alpha$.

Given an infinite cardinal κ we can define a class of κ-Borel sets using the cardinal κ in the same manner as we defined the class of Borel sets using the cardinal ω. The countable unions and the countable intersections are now going to be replaced by unions and intersections of length κ. However, only the sets $\Sigma_1^0(\kappa)$ and $\Pi_1^0(\kappa)$ from the first level of the κ-Borel hierarchy are going to be useful in the sequel. Thus, a set is $\Sigma_1^0(\kappa)$ if it can be obtained as a union of κ many internal sets and is $\Pi_1^0(\kappa)$ if it is an intersection of κ many internal sets.

The *finite projective hierarchy* of Σ_1^1 and Π_1^1 sets is defined as follows. Projections of Borel sets are called Σ_1^1 sets, and complements of Σ_1^1 sets are called Π_1^1 sets. Inductively, a Σ_{m+1}^1 set is the projection of a Π_m^1 set and complements of Σ_{m+1}^1 sets are Π_{m+1}^1 sets. In this paper we are mostly concerned with the first level of the finite projective hierarchy. It is known that every Σ_1^1 set which is at the same time Π_1^1 must already be Borel (Henson). In particular, every element of a sequence of disjoint Π_1^1 sets whose union is Σ_1^1 must already be a Borel set. For a detailed treatment of Descriptive Set Theory of Hyperfinite Sets the reader is referred to [5].

By a *graph* we mean a subset of the product of two sets X and Y. If these sets are internal, then a graph is *Borel* (Π_1^1 or Σ_1^1) if it is such as a subset of the product of X and Y. Given a graph $\Gamma \subseteq X \times Y$ and a point $x \in X$ we define the *Y-section* $\Gamma(x)$ of Γ at x as the set $\{y \in Y : (x, y) \in \Gamma\}$. The *domain* $\mathrm{dom}(\Gamma)$ of Γ is defined as the set of all x for which $\Gamma(x)$ is nonempty. Given two graphs Γ and Δ in $X \times Y$ and a measure μ in X we say that Γ and Δ are a.e. equal with respect to μ if the set $\{x \in X : \Gamma(x) \neq \Delta(x)\}$ has μ measure zero. If, in addition, Δ is an internal (e.g., Σ_1^0, Borel) graph, then we shall call Δ an *internal* (e.g., Σ_1^0, *Borel*) *lifting* of Γ. (One should stress at this point that the notion of lifting defined above is different from what is usually called a lifting in Nonstandard Analysis. I have decided to keep the same name here, despite a risk of possible confusion, in order to be consistent with the definitions in [9] where this notion of lifting was originally introduced.) The *restriction* $\Gamma|A$ of a graph Γ to a set A is the graph $\Gamma \cap (A \times Y)$.

A *uniformization* of a graph Γ is a partial function f whose graph is a subset of Γ and with the domain equal to the domain of Γ.

To every graph $\Gamma \subseteq X \times Y$ we can associate a partial *set function* f_Γ, which maps X into the internal power set of Y, by defining $f_\Gamma(x) = \Gamma(x)$, whenever $\Gamma(x)$ is internal and $\neq \varnothing$. Note that f_Γ is defined even if not all of the Y-sections of Γ are internal. If Y is hyperfinite, then the upper and the lower edges of Γ are defined as the compositions of the function f_Γ with the functions min and max respectively.

We assume that the reader is familiar with basic notions from Nonstandard Analysis and Descriptive Set Theory of Hyperfinite Sets [4], [5]. In all the statements our nonstandard universe is at least ω_1-saturated.

3. Results

Now we state and prove our main theorem which reveals the properties of Π_1^1 graphs, all of whose Y-sections are $\Sigma_1^0(\kappa)$ sets. The proof, in fact, uses an argument already exploited in [12, Proposition 2.4]. where the structure of Borel graphs with $\Pi_1^0(\kappa)$ ($\Sigma_1^0(\kappa)$) Y-sections was exhibited.

Theorem 1. *Let X and Y be internal sets, and let $\Gamma \subseteq X \times Y$ be a Π_1^1 graph, all of whose Y-sections are $\Sigma_1^0(\kappa)$ sets. Then the following are true.*

(a) *Γ can be represented as a countable union of restrictions of internal graphs to Π_1^1 sets.*

(b) *The domain of Γ is a Π_1^1 set.*

(c) *The sets $C = \{x \in X : \Gamma(x) \neq \varnothing$ and noninternal$\}$ and $D = \{x \in X : \Gamma(x) \neq \varnothing$ and internal$\}$ are in the least σ-algebra generated by Σ_1^1 and Π_1^1 sets.*

(d) *The graph of the set function associated with Γ is an intersection of a Σ_1^1 graph and a Π_1^1 graph. Thus, both the graph of the upper edge of Γ and the graph of the lower edge of Γ are intersections of a Σ_1^1 and a Π_1^1 graph.*

(e) *Given an internal, bounded measure* μ, Γ *possesses a* Σ_1^0, $L(\mu)$ *lifting. Also, keeping the notation from* (c), *the graph* $\Gamma|D$ *possesses an internal,* $L(\mu)$ *lifting. In particular, any* Π_1^1 *graph, all of whose* Y*-sections are internal sets, possesses an internal lifting and any* Π_1^1 *function is a.e. equal to an internal function.*

(f) Γ *possesses a* Π_1^1 *uniformization which can be represented as a countable union of restrictions of internal functions to disjoint* Π_1^1 *sets. In particular, every* Π_1^1 *function is a countable union of restrictions of internal functions to disjoint* Π_1^1 *sets. Also, if the domain of* Γ *is* Σ_1^1, *then* Γ *possesses a Borel uniformization. Therefore, any* Π_1^1 *function with* Σ_1^1 *domain must be Borel.*

Proof. (a) The proof of this part is essentially contained in the proof of Proposition 2.4. in [12]. It has been proved (Proposition 2.6, [12]) that, under the above assumptions, there exists a countable sequence Γ_n $(n \in \omega)$ of internal graphs in $X \times Y$ such that for every x the Y-section $\Gamma(x)$ is a union of sets of the form $\Gamma_n(x)$. After that it is easy to obtain the desired representation of Γ.

Let $P_n = \{x \in X : \Gamma_n(x) \neq \varnothing \wedge \Gamma_n(x) \subseteq \Gamma(x)\}$. It is clear that P_n is a Π_1^1 set as it is defined by the conjunction of an internal property $\Gamma_n(x) \neq \varnothing$ and by a Π_1^1 formula $(\forall y)[(x, y) \in \Gamma_n \rightarrow (x, y) \in \Gamma]$. We have

$$\Gamma = \bigcup_{n \in \omega} \Gamma_n | P_n. \tag{1}$$

(b) From the above representation we see that the domain of Γ is a countable union of Π_1^1 sets P_n and, therefore, is a Π_1^1 set.

(c) Let $\{\Gamma_n : n \in \omega\}$ be as in the proof of part (a). We can assume that $\{\Gamma_n : n \in \omega\}$ is closed with respect to taking finite unions and intersections. Therefore, $\Gamma(x)$ is going to be nonempty and noninternal if and only if it can be represented as a strictly increasing union of sets of the form $\Gamma_n(x)$. Therefore, $C = \{x \in X : \Gamma(x) \neq \varnothing \text{ and noninternal}\}$ is defined by the formula

$$\begin{aligned}(\exists n \in \omega)(\Gamma_n(x) \subseteq \Gamma(x) \ \wedge \ \Gamma_n(x) \neq \varnothing)& \\ \wedge \ (\forall k \in \omega)(\exists p \in \omega)\big[p > k \ \wedge \ [\Gamma_k(x) \subseteq \Gamma(x)& \\ \rightarrow (\Gamma_k(x) \subset \Gamma_p(x) \wedge \Gamma_p(x) \subseteq \Gamma(x))]\big].&\end{aligned}$$

Here $\Gamma_k(x) \subset \Gamma_p(x)$ is an internal property expressing strict inclusion. The first formula defines a Π_1^1 set. The second formula defines a set which is obtained by taking a countable union and a countable intersection of sets defined by the formula $p > k \ \wedge \ [\Gamma_k(x) \subseteq \Gamma(x) \rightarrow (\Gamma_k(x) \subset \Gamma_p(x) \wedge \Gamma_p(x) \subseteq \Gamma(x))$. The latter defines a union of a Σ_1^1 set and a Π_1^1 set. So D is in the σ-algebra generated by Σ_1^1 and Π_1^1 sets.

The set $D = \{x \in X : \Gamma(x) \text{ nonempty and internal}\}$ is now the intersection of the domain of Γ, a Π_1^1 set, and the complement of C. So D is also is in the σ-algebra generated by Σ_1^1 and a Π_1^1 set.

(d) Let f_n be the set function associated with the graph Γ_n. The graph of the set function f_Γ associated with Γ is defined as: $(x, A) \in f_\Gamma$ if and only if

$$(\exists n \in \omega)[(x, A) \in f_n | P_n] \quad \wedge \quad (\forall k \in \omega)(x \in P_k \rightarrow f_k(x) \subseteq A).$$

The first conjunct is Π_1^1 and the second is Σ_1^1. Taken together they define a function whose graph is an intersection of a Π_1^1 graph and a Σ_1^1 graph.

Suppose that Y is hyperfinite. The upper edge of Γ is defined as the composition of the function max and the function f_Γ. Following its definition above the graph h of the upper edge of Γ is defined as $(x, y) \in h$ if and only if

$$(\exists n \in \omega)[(x, y) \in \max \circ (f_n | P_n)] \quad \wedge \quad (\forall k \in \omega)(x \in P_k \rightarrow \max(f_k(x)) \leq y)].$$

It is easy to see that the composition of an internal and a Π_1^1 function is again Π_1^1. Thus, the first conjunct above is Π_1^1. The second is obviously Σ_1^1, so the graph of h is an intersection of a Σ_1^1 graph and a Π_1^1 graph.

We treat the lower edge similarly.

(e) Let μ be an internal, bounded measure in X. As Π_1^1 sets are always measurable and since μ is a bounded measure, for every P_n there exists a Σ_1^0 set $S_n \subseteq P_n$ with $L(\mu)(P_n \backslash S_n) = 0$. Let $H = \bigcup_{n \in \omega} \Gamma_n | S_n$. It is easy to see that $H = \Gamma$, $L(\mu)$-a.e. Indeed, if $H(x) \neq \Gamma(x)$, then for some n we must have $x \in P_n \backslash S_n$. So $\{x \in X : H(x) \neq \Gamma(x)\} \subseteq \bigcup_{n \in \omega} P_n \backslash S_n$ and the latter set is of measure zero. Therefore, H is a Σ_1^0 lifting of Γ.

To get an internal lifting of $\Gamma | D$ notice that D is $L(\mu)$ measurable. We can find a Σ_1^0 subset S of D with $L(\mu)(D \backslash S) = 0$ and with the graph $H|S$ having all of its Y-sections internal sets. Using ω_1-saturation we can then easily find an internal lifting of $H|S$ and thus of $\Gamma | D$.

(f) Using Reduction Theorem for Π_1^1 sets (see [5] for example), there exists a sequence $R_n \subseteq P_n$ of disjoint Π_1^1 sets with $\bigcup_{n \in \omega} R_n = \bigcup_{n \in \omega} P_n$. Now, we can take any internal uniformization g_n of Γ_n and observe that $\bigcup_{n \in \omega} g_n | R_n$ is a Π_1^1 uniformization of Γ of desired form. In addition, if the domain of Γ is Σ_1^1, then every R_n must be Borel. □

Remark 2. Suppose that Γ is a Π_1^1 graph in $X \times Y$. The proof of (a) of Theorem 1 then shows that if there exists a sequence of Borel graphs Γ_n $(n \in \omega)$ with internal Y-sections such that for every x $\Gamma(x)$ contains $\Gamma_n(x)$ for some n, with $\Gamma_n(x) \neq \varnothing$, then the domain of Γ must be Π_1^1. Indeed, if we define the sets P_n as in the proof of Theorem 1 (a), then P_n are still going to be Π_1^1. It is known (see [9] or [12]) that a Borel graph with internal Y-sections must have a Borel domain. Therefore, the property $\Gamma_n(x) \neq \varnothing$ in the definition of P_n is Borel and, thus, P_n is Π_1^1. By our assumption, $\operatorname{dom}(\Gamma) = \bigcup_{n \in \omega} P_n$, so the domain of Γ is Π_1^1.

So, by Theorem 1(f), two hyperfinite sets cannot be Π_1^1 bijective unless that bijection is already Borel. It would be interesting to give a sufficient and necessary condition for two true (i.e., non-Borel) Π_1^1 subsets P and Q of a hyperfinite set X that ensure the existence of a Π_1^1 bijection between them. For Borel P and Q it was shown in [8] that P and Q are Borel bijective if and only if P and Q have the same nonvanishing $L(\mu)$ measure for some counting, uniformly distributed internal measure μ. (Here, the word nonvanishing means that the measure of P and Q are not 0 or ∞.)

It is not true in general that a Π_1^1 graph with internal Y-sections can be represented as a countable union of restrictions of internal graph to disjoint Π_1^1 sets, i.e. our representation (1) in Theorem 1. is best possible. The key fact is that any graph that allows such a representation and has a Σ_1^1 domain must already be Borel because each P_n is, in this case, Borel.

Example 3. Let X be an internal set, $Y = \{1, 2\}$, and $P \subseteq X$ be a true Π^1_1 set (i.e., a Π^1_1 set which is not Borel). Let f be a function defined to be 1 on P and undefined otherwise. Let g be defined identically to be 2 on all of X. Define Γ to be the union of the graphs of f and g. We show that Γ is a Π^1_1 graph with internal Y-sections which cannot be represented as a countable union of restrictions of internal graphs to disjoint Π^1_1 sets. In addition, the set function associated with Γ as well as the lower edge of Γ are not Π^1_1.

It is clear that Γ is a Π^1_1 graph with all of its Y-sections internal sets. First note that the domain of Γ is Borel (in fact, internal). Now if Γ is a countable union of restrictions of internal graphs to disjoint Π^1_1 sets P_n, then each P_n would be Borel. Thus, Γ would be a Borel graph. This in turn means that the graph of f would also be Borel and therefore P is Borel. This is a contradiction.

If the set function f_Γ is Π^1_1, then it must be Borel because it has a Borel domain. It follows that Γ is Borel because $(x, y) \in \Gamma$ if and only if $(\exists A)((x, A) \in f_\Gamma \wedge y \in A))$ (a Σ^1_1 formula) and $(x, y) \notin \Gamma$ if and only if $(\exists A)((x, A) \notin f_\Gamma \wedge y \in A))$ (again a Σ^1_1 formula).

Similarly, if the lower edge of Γ is Π^1_1, then it must be Borel and again P would be Borel.

An easy variation of the above example also shows that, in addition, the answer to the open question 1 in [9] is negative. The question, motivated by a similar result for Σ^1_1 functions, asked if every Σ^1_1 graph in $X \times Y$, all of whose Y-sections are internal sets, possesses a complete Borel extension. That is, does there exist a Borel graph $\Delta \supseteq \Gamma$ with $\mathrm{dom}(\Delta) = X$ such that all of the Y-sections of Δ are internal and for every $x \in \mathrm{dom}(\Gamma)$, $\Delta(x) = \Gamma(x)$? If this were true for every Γ, then the set function associated with Γ would be Σ^1_1 because it is equal to the restriction of the set function f_Δ (which was proved to be Borel in [9]) to the domain of Γ. However, if in the above example we take P to be a true Σ^1_1 set instead of a true Π^1_1, then f_Γ would be Borel as its domain is Borel (a Σ^1_1 function with a Borel domain must be Borel). We conclude that Γ is Borel and obtain a contradiction as above.

Remark 4. Any Π^1_1 subset P of an internal set X can be generically expressed as the union of an increasing sequence B_α $(\alpha < \omega_1)$ of Borel sets such that every Σ^1_1 subset $S \subseteq P$ is contained in some B_β, $\beta < \omega_1$ (see [11]). The sets B_α are called constituents of P. Now when we know that the domain of any Π^1_1 function f is Π^1_1 we can ask for the connection between the constituents of f and the constituents of $\mathrm{dom}(f)$. In fact, it is easy to see that if f_α $(\alpha < \omega_1)$ are constituents of f then $\mathrm{dom}(f_\alpha)$, $(\alpha < \omega_1)$ are constituents of $\mathrm{dom}(f)$. Indeed, $\mathrm{dom}(f_\alpha)$ are Borel because the domains of Borel functions are always Borel. Given a Σ^1_1 subset S of $\mathrm{dom}(f)$ the function $f|S$ must Borel (Theorem 1(f)) and, thus, is a subset of some f_β. It follows that $S \subseteq \mathrm{dom}(f_\beta)$. Conversely, suppose that B_α $(\alpha < \omega_1)$ are constituents of $\mathrm{dom}(f)$. Then, again $f_\alpha = f|B_\alpha$ are Borel functions and, as one can easily see, they are constituents of f.

The following corollary is obtained by passing to complements.

Corollary 5. *Let Γ be a Σ^1_1 graph in $X \times Y$. Suppose that all of the Y-sections of Γ are $\Pi^0_1(\kappa)$ sets. Then the following hold.*

(a) Γ *can be represented as a monotone, countable intersection of the form*

$$\Gamma = \bigcap_{n\in\omega} (\Gamma_1^n|S_1^n \cup \cdots \cup \Gamma_{k_n}^n|S_{k_n}^n) \tag{2}$$

where $\Gamma_1^n, \ldots, \Gamma_{k_n}^n$ *are internal graphs and* $S_1^n, \ldots, S_{k_n}^n$ Σ_1^1 *sets.*

(b) *The sets* $C = \{x \in X : \Gamma(x) \neq \varnothing$ *and noninternal*$\}$ *and* $D = \{x \in X : \Gamma(x) \neq \varnothing$ *and internal*$\}$ *are in the least* σ*-algebra generated by* Σ_1^1 *and* Π_1^1 *sets.*

(c) *For every bounded internal measure* μ, Γ *has* Π_1^0, $L(\mu)$ *lifting and* $\Gamma|D$ *has an internal lifting.*

(d) *The graph of the set function associated with* Γ *is an intersection of a* Σ_1^1 *graph and a* Π_1^1 *graph.*

Proof. We can assume that $\Gamma(x) \neq Y$, by enlarging Y if necessary.

(a) The complement Γ^c of Γ can be represented as (1). In other words

$$\Gamma^c = \bigcup_{n\in\omega} \Gamma_n|P_n$$

for some internal graphs Γ_n and some Π_1^1 sets P_n. Therefore, Γ can be expressed as

$$\Gamma = \bigcap_{n\in\omega}(\Gamma_n \cap (P_n \times Y))^c = \bigcap_{n\in\omega}(\Gamma_n^c \cup (P_n^c \times Y)).$$

Now, the finite intersection of sets of the form $\Gamma_n^c \cup (P_n^c \times Y)$ is easily seen to be a finite union of restrictions of internal graphs to Σ_1^1 sets and the representation (2) follows.

Part (b) follows again by looking at the complement of Γ. Part (c) follows from the above fact that Γ can be represented as a countable intersection of finite unions of graphs, each of which is a restriction of an internal graph to a Σ_1^1 set. The latter graphs obviously have internal liftings so that, at the end, Γ has a Π_1^0 lifting. The same lifting can be used to obtain an internal lifting of $\Gamma|D$ in an obvious manner.

(d) Let P_n and f_n be as in the proof of parts (a) and (d) of Theorem 1 applied to the complement of Γ. Then the graph of f_Γ can be defined as $(x, A) \in f_\Gamma$ if and only if

$$x \in \mathrm{dom}(\Gamma) \ \wedge\ (\forall y)((x, y) \in \Gamma \to y \in A) \ \wedge\ (\forall n \in \omega)[x \in P_n \to A \cap f_n(x) = \varnothing].$$

The property $x \in \mathrm{dom}(\Gamma)$ is Σ_1^1 and ensures that $\Gamma(x)$ is not empty. The formula $(\forall y)((x, y) \in \Gamma \to y \in A)$ asserts that $\Gamma(x) \subseteq A$ and is Π_1^1. Finally, $(\forall n \in \omega)[x \in P_n \to A \cap \varphi_n(x) = \varnothing]$ is a Σ_1^1 formula expressing $A \subseteq \Gamma(x)$. □

It is natural to ask if the structural results of Corollary 5 can be strengthened in the case when all of the Y-sections of Γ are internal rather than $\Pi_1^0(\kappa)$ sets. For example, is it true that a Σ_1^1 graph Γ, all of whose Y sections are internal sets, has a representation (1) with P_n being Σ_1^1 sets? It is known that any Σ_1^1 function f can be extended to a Borel function [10] and that any Borel function is a countable union of restrictions of internal functions to Borel sets [9]. It then follows, by intersecting with the domain of f, that any Σ_1^1 function is a countable union of restrictions of internal functions to disjoint Σ_1^1 sets. The same is true for Σ_1^1 graphs, all of whose Y-sections are of cardinality $\leq n$ for a fixed, finite integer n or, more generally, when the set function associated with Γ is itself Σ_1^1.

In the same way, does a Σ_1^1 graph $\Gamma \subseteq X \times Y$, all of whose Y sections are internal $(\Pi_1^0(\kappa))$ sets, possess a complete Σ_1^1 extension? In other words, does there exist a Σ_1^1 graph $\Delta \supseteq \Gamma$, all of whose Y sections are internal $(\Pi_1^0(\kappa))$ sets, such that $\operatorname{dom}(\Delta) = X$ and $\Delta(x) = \Gamma(x)$ for every $x \in \operatorname{dom}(\Gamma)$? We know that the answer is positive, in one special case, when Γ is a finite union of restrictions of internal $(\Pi_1^0(\kappa))$ graphs to Σ_1^1 sets.

Next, we give a partial answer to Problem 1.5 in [3]. A graph $\Gamma \subseteq X \times Y$ is said to be *universal* for *internal* subsets of Y if every internal subset of Y can be obtained as a Y section of Γ. The internal power set of Y is denoted as 2^Y.

Corollary 6. *Let X and Y be hyperfinite sets, and let Γ be a Π_1^1 (Σ_1^1) graph in $X \times Y$ universal for all internal subsets of Y. Then $|X|/2^{|Y|} \geq$ or ≈ 1.*

Proof. We can only concentrate on Π_1^1 graphs because if Γ is universal for internal subsets of Y, then its complement is also universal for internal subsets of Y.

We first prove that there exists a Π_1^1 graph $H \subseteq \Gamma$ universal for internal subsets of Y and with all of its Y sections Σ_1^0 sets. As in the proof of Theorem 1, there exists a sequence Γ_n $(n \in \omega)$ of internal graphs in $X \times Y$ such that for every x if $\Gamma(x)$ is internal, then there exists n such that $\Gamma(x) = \Gamma_n(x)$. Let $P_n = \{x \in X : \Gamma_n(x) \subseteq \Gamma(x)\}$. Note that P_n are all Π_1^1 sets. Now, define $H = \bigcup_{n\in\omega} \Gamma_n$. It is clear that H is a Π_1^1 graph with Σ_1^0 Y-sections, universal for internal subsets of Y.

Define an internal measure μ by setting $\mu(A) = |A|/2^{|Y|}$ for every hyperfinite A. If X has unbounded μ measure, then we are done. If the measure of X is finite, then we proceed as follows. Let $D = \{x \in X : H(x) \text{ internal and } \neq \varnothing\}$. Then D is $L(\mu)$ measurable and $H|D$ has an internal lifting G. Let f_n be the set functions associated with $\Gamma_n|P_n$, and let f_G be the set function associated with G. We prove that the range of f_G is equal almost to all of the set 2^Y. Indeed, if a set B is not in the range of f_G, then it must be of the form $f_n(x)$ for some n and some $x \in D \triangle \operatorname{dom}(G)$. But f_n maps measure-zero sets onto measure zero sets, being a restriction of an internal function to a measurable set. Therefore, the complement of the range of f_G in 2^Y is of $L(\mu)$ measure zero. Now we are done because $L(\mu)(2^Y) = 1 = L(\mu)(\text{range}(f_G)) \leq L(\mu)(X) \approx |X|/2^{|Y|}$. □

At the end we give some application of Theorem 1 and its corollaries to the standard Descriptive Set Theory. We call a set in arbitrary topological space X a F_σ set if it is a countable union of closed sets in X. The standard part map associated with the topology in X is denoted as st_X, whereas the standard part map in the product $X \times Y$ of two spaces X and Y is denoted simply as st, if there is no danger of confusion. The set of near standard elements is denoted as n.s.$({}^*X)$. Given a Borel measure μ in X we say that the standard part map $\operatorname{st}_X : \text{n.s.}({}^*X) \to X$ is measure preserving if for every set $M \subseteq X$, we have $\operatorname{st}_X^{-1}(M)$ is $L({}^*\mu)$ measurable if and only if M is μ measurable and $\mu(M) = L({}^*\mu)(\operatorname{st}_X^{-1}(M))$.

Theorem 7. *Let X and Y be Hausdorff topological spaces, and let $G \subseteq X \times Y$ be a graph such that*

(i) $\mathrm{st}^{-1}(G)$ *is a* Π_1^1 *subset of* $^*X \times {}^*Y$

in some κ^+-saturated nonstandard universe, where κ is greater than the maximum of the cardinalities of the basis in X and Y. Suppose that

(ii) $\mathrm{st}_Y^{-1}(G(x))$ *is* $\Sigma_1^0(\kappa)$ *for every* x.

Suppose that μ is a complete, Borel probability measure in X such that

(iii) $\mathrm{st}_X : n.s.(^*X) \to X$ *is measure preserving.*

Then, the following hold.

(a) $\mathrm{st}_X^{-1}(\mathrm{dom}(G))$ *is a* Π_1^1 *set.*

(b) *There exists an F_σ graph H in $X \times Y$ such that $H = G$, μ-a.e. If X and Y are regular, then H can be chosen to be σ-compact.*

(c) *If Y is regular, then the set $C = \{x \in X : G(x)$ compact and nonempty$\}$ is μ measurable and $G|C$ is a.e. equal to a compact graph.*

Proof. Define $\Gamma = \mathrm{st}^{-1}(G)$. It is easy to see that all of the Y-sections of Γ are $\Sigma_1^0(\kappa)$. Indeed, let $\alpha \in {}^*X$ be in the domain of Γ. Suppose that $\mathrm{st}_X(\alpha) = x$. We have $\Gamma(\alpha) = [m(x) \times \mathrm{st}_Y^{-1}(G(x))] \cap (\alpha \times {}^*Y) = \mathrm{st}_Y^{-1}(G(x))$. This is a $\Sigma_1^0(\kappa)$ set.

(a) By Theorem 1 (b), the domain of Γ is Π_1^1. Now, we always have $\mathrm{st}_X^{-1}(\mathrm{dom}(G))$ equal to $\mathrm{dom}(\mathrm{st}_X^{-1}(G))$ which is, in turn, equal to the domain of Γ and we are done.

(b) By Theorem 1 (e) there exists a Σ_1^0, $L(^*\mu)$ lifting $F \subseteq \Gamma$ of Γ. As the standard part image of an internal set in any topological space is closed, $\mathrm{st}(F)$ is a countable union of closed sets in $X \times Y$ μ-a.e. equal to G. Also, if X and Y are regular, then its product is regular and in any regular topological space the standard part image of any internal set all of whose points are near standard is compact.

(c) We prove that

$$\{\alpha \in {}^*X : \Gamma(\alpha) \text{ internal and } \neq \varnothing\} = \mathrm{st}^{-1}(\{x \in X : G(x) \text{ compact and } \neq \varnothing\}).$$

Indeed, let $\Gamma(\alpha)$ be internal and nonempty. As every point of $\Gamma(\alpha)$ is near standard and Y regular, $\mathrm{st}_Y(\Gamma(\alpha))$ is compact. Also, $\mathrm{st}_Y(\Gamma(\alpha)) = G(x)$, where $x = \mathrm{st}_X(\alpha)$. We showed that $\alpha \in \mathrm{st}^{-1}(\{x \in X : G(x)$ compact and $\neq \varnothing\})$.

Conversely, suppose that $\alpha \in \mathrm{st}^{-1}(\{x \in X : G(x)$ compact and $\neq \varnothing\})$. Let x be such that $\mathrm{st}_X(\alpha) = x$ and $G(x)$ compact. Now, $\Gamma(\alpha) = \mathrm{st}_Y^{-1}(G(x))$ is a $\Pi_1^0(\kappa)$ set because $G(x)$ is compact. On the other hand, by our assumption, $\mathrm{st}_Y^{-1}(G(x))$ is $\Sigma_1^0(\kappa)$. By κ^+ saturation $\mathrm{st}_Y^{-1}(G(x))$ must be internal. This establishes the above equality.

Now, the set $\{\alpha \in {}^*X : \Gamma(\alpha)$ internal and $\neq \varnothing\}$ is a member of the least σ-algebra containing Π_1^1 and Σ_1^1 sets and therefore is $L(^*\mu)$-measurable. As st is μ-measure-preserving the set $\mathrm{st}^{-1}(\{x \in X : G(x)$ compact and $\neq \varnothing\})$ is μ measurable.

To prove the second part of (c) notice that by Theorem 1 (e) $\Gamma|\{\alpha \in {}^*X : \Gamma(\alpha)$ internal and $\neq \varnothing\}$ has an internal lifting. The standard part image of that lifting is a compact lifting of $G|C$. □

In the same vein we prove the following extension of Proposition 3.9 from [12].

Theorem 8. *Let X and Y be Hausdorff topological spaces, and let $G \subseteq X \times Y$ be a graph such that*

(i) $\mathrm{st}^{-1}(G)$ *is a* Σ_1^1 *subset of* $^*X \times {}^*Y$

in some κ^+*-saturated nonstandard universe, where* κ *is greater than the maximum of the cardinalities of the basis in* X *and* Y. *Suppose that*

(ii) $\mathrm{st}_Y^{-1}(G(x))$ *is* $\Pi_1^0(\kappa)$ *for every* x.

Suppose that μ *is a complete, Borel probability measure in* X *such that*

(iii) $\mathrm{st}_X : \mathrm{n.s.}(^*X) \to X$ *is measure preserving.*

Then, for every $\epsilon > 0$ *there exists a closed graph* H_ϵ *in* $X \times Y$ *such that* $H_\epsilon \subseteq G$; $\mathrm{dom}(H_\epsilon)$ *is* μ *measurable;* $\mu(\mathrm{dom}(H_\epsilon)) \geq \mathrm{dom}(G) - \epsilon$, *and for all* $x \in \mathrm{dom}(H_\epsilon)$, $H_\epsilon(x) = G(x)$.

If X *and* Y *are regular, then* H_ϵ *can be chosen to be compact.*

Proof. The proof follows the same lines as the proof of Proposition 3.9. in [12] once we realize that, following Corollary 5 (c) $\mathrm{st}^{-1}(G)$ possesses a Π_1^0 lifting. □

We shall now specify those topological spaces X and Y and those graphs G for which conditions (i), (ii) and (iii) of Theorems 7 and 8 are fulfilled. For that purpose we restrict ourselves only to completely regular, Hausdorff spaces. A completely regular, Hausdorff space X is called *proper K-Lusin* if it is a Baire set in each of its compactifications. A subset S of X is called *proper K-analytic* if it can be obtained by applying the Souslin operation to Baire sets in any compactification Y of X. In this paper the complements of proper K-analytic sets are called *co-proper K-analytic* (for more details see [7]).

Ward Henson proved (see [2]) that $S \subseteq X$ is proper K-Lusin if and only if $\mathrm{st}_X^{-1}(S)$ is Borel in $^*X$ and in particular, X is proper K-Lusin if and only if the set $\mathrm{n.s.}(^*X)$ of near-standard points in $^*X$ is Borel in $^*X$. Among metrizable spaces, proper K-Lusin spaces include separable absolute Borel spaces (a space is absolute Borel if it is Borel in each of its metrizable extensions) and in particular every Polish space (complete, separable metric space).

Also, the standard part image of a Σ_1^1 set in $^*X$ is always proper K-analytic [2]. It follows that if $\mathrm{st}_X^{-1}(S)$ is Π_1^1 in $^*X$, then the complement $X \backslash S$ of S in X is the st_X image of the complement of a Π_1^1 set $\mathrm{st}_X^{-1}(S)$ in $^*X$. Thus, S is co-proper K-analytic. However, the converse seems not to be true for arbitrary completely regular X. Recently, Herman Render [6] proved, answering an old conjecture of Henson, that for a subset S of X, $\mathrm{st}_X^{-1}(S)$ is Σ_1^1 in $^*X$ if and only if S is proper K-analytic. Thus, if X is proper K-Lusin space, then $\mathrm{st}_X^{-1}(S)$ is Π_1^1 in $^*X$ if and only if S is co-proper K-analytic.

It is shown in [12] that for $S \subseteq X$, $\mathrm{st}^{-1}(S)$ is $\Pi_1^0(\kappa)$, where κ is greater than the cardinality of the base in X if and only if S is compact. Also, it is easy to see that if X is a locally compact topological space, then S is open in X if and only if $\mathrm{st}_X^{-1}(S)$ is $\Sigma_1^0(\kappa)$ in $^*X$.

Condition (iii) is satisfied if μ is taken to be a Radon measure in X, as was shown in [1]. (A Borel measure μ is called *Radon* if

$$\mu(B) = \sup\{\mu(C) : C \subseteq B, C \text{ compact}\} = \inf\{\mu(O) : O \supseteq B, O \text{ open}\}.)$$

Putting all the above together we obtain

Corollary 9. *Let X and Y be proper K-Lusin spaces with Y being, in addition, locally compact. Let $G \subseteq X \times Y$ be a co-proper K-analytic graph, all of whose Y-sections are open sets. Suppose that μ is a Radon probability measure in X. Then*

(a) *the domain of G is a co-proper K-analytic set;*
(b) *there exists an σ-compact graph H in $X \times Y$ such that $H = G$, μ-a.e.;*
(c) *the set $\{x \in X : G(x)$ compact and $\neq \varnothing\}$ is μ-measurable.*

Proof. Condition (i) of Theorem 7 is fulfilled because the product of two proper K-Lusin spaces is again proper K-Lusin (as one easily can check by using the above described nonstandard characterization of being proper K-Lusin for completely regular spaces). Condition (ii) is satisfied because Y is locally compact, while condition (iii) is always satisfied for Radon measures. It follows that $\mathrm{st}_X^{-1}(\mathrm{dom}(G))$ is Π_1^1 in $^*X$, so $\mathrm{dom}(G)$ is co-proper K-analytic.

Parts (b) and (c) follow from the corresponding parts of Theorem 7. □

In the same manner, but this time using Theorem 8, we obtain

Corollary 10. *Let X and Y be completely regular topological space. Let $G \subseteq X \times Y$ be a proper K-analytic graph, all of whose Y-sections are compact sets. Suppose that μ is a Radon probability measure in X. Then for every $\epsilon > 0$ there exists an compact graph H_ϵ in $X \times Y$ such that $H_\epsilon \subseteq G$; $\mu(\mathrm{dom}(H_\epsilon)) \geq \mathrm{dom}(G) - \epsilon$, and for all $x \in \mathrm{dom}(H_\epsilon)$, $H_\epsilon(x) = G(x)$.* □

Example 11. Let E and F be Polish spaces, and let f be a functions whose graph is a true Π_1^1 subset of $E \times F$. We identify functions with their graphs and set $X = {}^*E$, $Y = {}^*F$. Suppose that the domain of f is Borel. Then $\Gamma = \mathrm{st}_{E\times F}^{-1}(f)$ is an example of a Π_1^1 graph with Π_1^0 Y-sections which cannot be expressed as a countable intersection of Π_1^1 graphs with internal Y-sections.

Indeed, let $\Gamma = \bigcap_{n\in\omega} \Gamma_n$ with Γ_n being Π_1^1 graphs with internal Y-sections. We may suppose that $\mathrm{dom}(\Gamma_n) = \mathrm{dom}(\Gamma)$. By Theorem 1(f), Γ_n possesses a Borel uniformization g_n. Let $G_n = \mathrm{st}_{X\times Y}(g_n)$. Then G_n is a Σ_1^1 subset of $E \times F$ with $\mathrm{dom}(G_n) = \mathrm{dom}(f)$. Graphs G_n have the property that for every $x \in \mathrm{dom}(f)$ and every $\epsilon > 0$ there exists n such that $G_n(x)$ is at distance less than ϵ from $f(x)$. Therefore, f is defined as $(x, y) \in f$ if and only if $(\forall n \in \omega)(\exists y_n)[(x, y_n) \in G_n \wedge d(y, y_n) < 1/n]$ where d is the distance function in F. This formula defines a Σ_1^1 function, so f is Borel—a contradiction.

There exists a standard example of a closed subset C of $\omega^\omega \times \omega^\omega$ with the domain equal to ω^ω which does not possess a Borel uniformization. By the Kondo-Addison uniformization theorem, there exists a Π_1^1 uniformization of C whose graph, therefore, is not Borel.

Another example of a true Π_1^1 function with a Borel domain arising in "real" mathematics is communicated to me by Professor Kechris. Let G be the set of all permutations of integers understood as a Polish group, and let X be the space of all closed subgroups of G. We consider X to be endowed with Effros Borel structure. Then the function $H \mapsto \{p \in G : pHp^{-1} = H\}$ which sends a closed subgroup to its normalizer is a true Π_1^1 function with the domain equal to X.

References

1. R.A. Anderson, *Star-finite representation of measure spaces*, Trans. Amer. Math. Soc. **271** (1982), 667–687.
2. C.W. Henson, *Analytic sets, Baire sets and the standard part map*, Canad. J. Math. **31** (1979), 663–672.
3. C.W. Henson and D. Ross, *Analytic mappings on hyperfinite sets*, Proc. Amer. Math. Soc. **118** (1993), 587–596.
4. A.E. Hurd and P.A. Loeb, *An introduction to nonstandard real analysis*, Academic Press, New York, 1985.
5. H.J. Keisler, K. Kunen, A. Miller, and S. Leth, *Descriptive set theory over hyperfinite sets*, J. Symbolic Logic **54** (1989), 1167–1180.
6. H. Render, private communication.
7. C.A. Rogers, *Analytic sets*, Academic Press, New York, 1980.
8. B. Živaljević, *Some results about Borel sets in descriptive set theory of hyperfinite sets*, J. Symbolic Logic **55** (1990), 604–614.
9. ———, *The structure of graphs all of whose Y-sections are internal sets*, J. Symbolic Logic **56** (1991), 50–66.
10. ———, *Every Borel function is monotone Borel*, Ann. Pure Appl. Logic **54** (1991), 87–99.
11. ———, *Lusin-Sierpinski index for the internal sets*, J. Symbolic Logic **57** (1992), 172–178.
12. ———, *Graphs with $\Pi^0_1(\kappa)$ Y-sections*, Arch. Math. Logic **32** (1993), 259–273.

Department of Computer Science, The University of Illinois at Urbana - Champaign, Urbana, Illinois 61801

E-mail address: `zivaljev@cs.uiuc.edu`